Capítulo 06

Topological Entropy, Pressure, and Ergodicity: New Theoretical Frameworks for Complex Dynamical Systems and Turbulent Flows

Entropia Topológica, Pressão e Ergodicidade: Novos Referenciais Teóricos para Sistemas Dinâmicos Complexos e Fluxos Turbulentos

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Abstract: This work proposes an investigation of the theory of dynamical systems, focusing on the analysis of entropy in its classical and topological manifestations. Initially, the fundamental concepts of dynamical systems theory are presented, with an emphasis on topological dynamical systems (TDS). Next, definitions of discrete topological entropy and its relationship to dynamical systems are discussed, followed by the introduction of topological entropy pressure as a weighted form of topological entropy. Subsequently, some applications of topological entropy in dynamic systems are examined, highlighting its role in the analysis of chaotic systems and in ergodic theory. Furthermore, we present a new theory, called Topological Ergodic Entropy Theory (TEET), which offers an innovative approach to the analysis of ergodic dynamical systems. Finally, the Ergodic Theory of Turbulent Flow (ETTF) is introduced, exploring the relationship between topological entropy and the ergodic properties of dynamic systems governed by the Navier-Stokes equations. The results presented contribute to a deeper understanding of the complexity of dynamical systems and their applications in various areas of mathematics and physics. The investigation of topological entropy and its applications in dynamical systems offers new insights into the chaotic and stochastic behavior of these systems, while the introduction of new theories, such as ETTF, provides a new perspective for the analysis and modeling of turbulent phenomena.

Keywords: entropy; topological pressure; ergodicity; turbulence in fluids; complex dynamic systems.

Resumo: Este trabalho propõe uma investigação sobre a teoria dos sistemas dinâmicos, com foco na análise da entropia em suas manifestações clássica e topológica. Inicialmente, são apresentados os conceitos fundamentais da teoria dos sistemas dinâmicos, com ênfase nos sistemas dinâmicos topológicos (TDS). Em seguida, discutem-se as definições de entropia topológica discreta e sua relação com os sistemas dinâmicos, seguida da introdução da pressão da entropia topológica como uma forma ponderada da entropia topológica. Posteriormente, são examinadas algumas aplicações da entropia topológica em sistemas dinâmicos, destacando seu papel na análise de sistemas caóticos e na teoria ergódica. Além disso, apresenta-se uma nova teoria, denominada Teoria da Entropia Ergodicamente Topológica (TEET), que oferece uma abordagem inovadora para a análise de sistemas dinâmicos ergódicos. Por fim, é introduzida a Teoria Ergodicamente Topológica do Escoamento Turbulento (ETTF), explorando a relação entre a entropia topológica e as propriedades ergódicas de sistemas dinâmicos governados pelas equações de Navier–Stokes. Os resultados apresentados contribuem para uma compreensão mais aprofundada da complexidade dos sistemas dinâmicos e de suas aplicações em diversas áreas da

DOI: 10.47573/aya.5379.3.51.6

Matemática e da Física. A investigação da entropia topológica e de suas aplicações em sistemas dinâmicos oferece novas perspectivas sobre o comportamento caótico e estocástico desses sistemas, enquanto a introdução de novas teorias, como a ETTF, fornece uma nova abordagem para a análise e modelagem de fenômenos turbulentos.

Palavras-chave: entropia; pressão topológica; ergodicidade; turbulência em fluidos; sistemas dinâmicos complexos.

1. INTRODUCTION

The word "entropy" was coined in 1865 by the German physicist and mathematician Rudolf Clausius, one of the pioneers of Thermodynamics. In the theory of systems in thermodynamic equilibrium, entropy quantifies the degree of "disorder" present in the system, representing a fundamental measure of its randomness and energy distribution. The origin of the word dates back to the Greek word 'entropia', which means "turning towards" (en: in; tropo: transformation), signifying 'measure of the disorder of a system '.

Sinai (2009) developed the concept of entropy for a system in Ergodic Theory (now referred to as 'Kolmogorov-Sinai entropy'). The authors, Adler *et al.* (1965), as pioneers in the notion of topological entropy, devised a method to assess its 'extent' by assigning a numerical value to their novel concepts and theories.

In this research, we present the concept of entropy from both a theoretical measure and a topological aspect, along with their prerequisites. Unfortunately, it was not feasible to cover all topics due to their extent, but it was possible to introduce an important topic of interest in the present work, as mentioned in its title. The references used are mentioned. However, their epitomes are: Viana & Oliveira (2016); Walters (2000); Einsiedler (2011) and Glasner (2003).

Mathematically presenting the optimal form of results, despite the challenges posed by the topic and its respective demonstrations, provides the necessary framework for understanding the subject matter of interest.

2. DYNAMIC SYSTEMS

The theory of dynamic systems encompasses the study of qualitative properties of group operations in spaces endowed with specific structures. In this study, our primary focus lies on operations via homeomorphisms in compact metric spaces, which possess an additional structure of invariant Borel probability measure under the operation in question.

2.1 Topological Dynamic System – TDS

A Topological Dynamical System (TDS) is understood as a pair (X,T), where X is a compact metric space and $T : X \to X$ is a self-homeomorphism. In some theorems, we also emphasize the properties of T, such as being surjective. In the references, the main definition of a TDS is a pair (X,T), where X is a compact metric space and $T : X \to X$ is a continuous map. Additionally, in Furstenberg, H. (1967), the author uses the term 'flow' for a TDS; and employs the term 'bilateral' for a TDS with a homeomorphism$T : X \to X$. However, in this work, a TDS was considered together with a self-homeomorphism.

Definition 2.1.1. For an SDT (X,T) and open sets U,V⊆X. *And still, with* $N(U,V) :=\{n\in\mathbb{N} | T^n U\cap V\neq\emptyset\}$:

- An SDT (X,T) is transitive if for each open set $U,V\subseteq X: N(U,V)\neq\emptyset$;
- An SDT (X,T) is mixing weakly if, for each open set $U_1,V_1,U_2,V_2\subseteq X, N(U_1,V_1)\cap N(U_2,V_2)\neq\emptyset$;
- An SDT (X,T) is mixed or mixed strongly if, for each open set U,V⊆X, there is $n_0 \in \mathbb{N}$, such that $\{n_0 \in \mathbb{N} | n > n_0\} \subseteq N(U,V)$.

Proposition 2.1.2. An SDT (X,T) is transitive if and only if, $\cup_{n\in\mathbb{N}} \Delta_n \subseteq X \times X$ it is dense, where $\Delta_n := \{(x, T^n x): x \in X\}$.

Proof. If (X,T) is transitive, for each open neighborhood $U \times V \subseteq X \times X \; of \, (x,y) \in X \times X$, since U is a neighbor of x, and V is a neighbor of y, there is $n_0 \in \mathbb{N}$ such that $T^n U \cap V \neq \emptyset$. Then, $(U \times T^n U) \cap (U \times V) \neq \emptyset$.Therefore,

$$(U \times V) \cap \bigcup_{n\in\mathbb{N}} \Delta_n \neq \emptyset.$$

In other words, $\cup_{n\in\mathbb{N}} \Delta_n$ is a dense subset of $X \times X$. On the other hand, if $\cup_{n\in\mathbb{N}} \Delta_n$ is a set of X, for each open set $U, V \subseteq X$, there is $n \in \mathbb{N}$ such that $(U \times V) \cap \Delta_n \neq \emptyset$, so that there is $(z, T^n z) \in U \times V$ Therefore,

$$T^n z \in T^n U \cap V \neq \emptyset.$$

3. TOPOLOGICAL ENTROPY

If we have a set X, we denote by C_X the set of all finite covers of X . If $\mathcal{U} \in C_X, \mathcal{N}(\mathcal{U})$, is denoted as the minimum cardinality of subcovers of $\mathcal{U}: \mathcal{N}(\mathcal{U}) := \min\{\#\mathcal{V} \in C_X, \mathcal{V} \subseteq \mathcal{U}\}$. Now consider a transformation $T: X \to X$. For a given integer number $M \leq N$ and $\mathcal{U} \in C_X$, then let $\mathcal{U}_M^N := \bigvee_{n=M}^N T^{-n}(\mathcal{U})$.

Definition 3.1. (Discrete Entropy). The discrete entropy of $\mathcal{U}$ with regards to $T: X \to X$is defined by

$$h_c(\mathcal{U}, T) := \lim_{n\to\infty} \frac{1}{n} \log\left(\mathcal{N}(\mathcal{U}_0^{n-1})\right). \tag{1}$$

Example 3.1.1. Let X be a set and $P \in C_X$ a partition of X, and considering the transformation $X \to X$, we have to $\mathcal{N}(P) = \#P$ and $P_0^n = P$, for each $n \in \mathbb{Z}$. So, for (3.1.), it is valid to say that $h_c(P, X \to X) := \lim_{n\to\infty} \frac{1}{n} \log(\#P) = 0$.

Proposition 3.1.1. The limit defined in (3.1) always exists.

Proof. Let $a_m := \mathcal{N}(\mathcal{U}_0^{n-1})$ be we will prove that $a_{m+n} < a_m a_n$. Initially, note that

$$\mathcal{U}_0^{n-1} = \bigvee_{i=0}^{m+n-1} T^{-i}\mathcal{U}$$

$$= \left(\bigvee_{i=0}^{m-1} T^{-i}\mathcal{U}\right) \bigvee \left(\bigvee_{i=0}^{m+n-1} T^{-i}\mathcal{U}\right)$$

$$= \left(\bigvee_{i=0}^{m-1} T^{-i}\mathcal{U}\right) \bigvee T^{-m}\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}\right)$$

$$= \mathcal{U}_0^{m-1} \bigvee T^{-m}\mathcal{U}_0^{n-1}.$$

So, if U_m is a cover of $\mathcal{U}_0^{m-1}$ and U_n is a cover of $\mathcal{U}_0^{n-1}$, both with minimum cardinality, it is valid that

$$U_m \bigvee T^{-m}U_n$$

is a cover of $\mathcal{U}_0^{m+n-1}$. So,

$$\#\left(U_m \bigvee T^{-m}U_n\right) = a_m a_n.$$

Therefore, we conclude that

$$a_{m+n} \leq a_m a_n.$$

4. TOPOLOGICAL ENTROPY PRESSURE

Pressure $P(T,\phi)$ is a weighted version of topological entropy $h_{top}(T)$, where the weights are determined with the continuous function $\phi: X \to \mathbb{R}$, where we call it the “potential function”. In some cases, we will see that

$$P(T,\phi) = h_{top}(T),$$

where $\phi: X \to \mathbb{R}$ is a null function.

For all $n \geq 1$, we have

$$\begin{cases} \phi_n : X \to R \\ \phi_n(x) = \sum_{i=0}^{n-1} \phi \circ T^i(x). \end{cases}$$

Now, for all $\alpha \in C_X^0$, we have:

$$P_n(T, \phi, \alpha) := \inf\{\sum_{U \in \gamma} \sup e^{\phi_n(x)} | \gamma \in C_X^0, \succcurlyeq \alpha_0^n\}. \quad (2)$$

Proposition 4.1.1. It is valid that $\log P_{n+m}(T, \phi, \alpha) \leq \log P_n(T, \phi, \alpha) + \log P_m(T, \phi, \alpha)$.

Proof. For simplicity, let P_n stands for $P_n(T, \phi, \alpha)$ and $\mathcal{U} \overset{cover}{\subseteq} \mathcal{V}$ denotes that $\mathcal{U}$ is a subcover de $\mathcal{V}$.

Since

$$\phi_{n+m} = \sum_{i=0}^{n+m-1} \phi \circ T^i = \phi_n + \sum_{i=n}^{m+n-1} \phi \circ T^i = \phi_n + \left(\sum_{i=0}^{m-1} \phi \circ T^i\right) \circ T^n = \phi_n + \phi_m \circ T^n,$$

we have

$$P_{n+m} = \inf\left\{\sum_{U \in \gamma} \sup \exp\left(\phi_{n+m}(x) \,\middle|\, \gamma \overset{cover}{\subseteq} \alpha_0^{n+m}\right)\right\}$$

$$= \inf\left\{\sum_{U \in \gamma} \sup \exp\left(\phi_{n+m}(x) \circ T^n(x) \,\middle|\, \gamma \overset{cover}{\subseteq} \alpha_0^{n+m}\right)\right\}$$

$$= \inf\left\{\sum_{U \in \gamma} \sup \exp(\phi_n(x) \circ T^n(x)) \exp \phi_m(x) \circ T^n(x) \,\middle|\, \gamma \overset{cover}{\subseteq} \alpha_0^{n+m}\right\}$$

$$\leq \inf\left\{\sum_{U \in \gamma} \sup \exp(\phi_n(x)) \sum_{U \in \gamma} \sup \exp(\phi_m(x) \circ T^n(x)) \,\middle|\, \gamma \overset{cover}{\subseteq} \alpha_0^{n+m}\right\}$$

$$= \inf\left\{\sum_{U \in \gamma} \sup \exp(\phi_n(x)) \,\middle|\, \gamma \overset{cover}{\subseteq} \alpha_0^{n+m}\right\} \inf\left\{\sum_{U \in \gamma} \sup \exp(\phi_m(x) \circ T^n(x)) \,\middle|\, \gamma \overset{cover}{\subseteq} \alpha_0^{n+m}\right\}$$

$$\leq \inf\left\{\sum_{U \in \gamma} \sup \exp(\phi_n(x)) \,\middle|\, \gamma \overset{cover}{\subseteq} \alpha_0^{n+m}\right\} \inf\left\{\sum_{U \in \gamma} \sup \exp(\phi_m(y)) \,\middle|\, \gamma \overset{cover}{\subseteq} \alpha_{-n}^{m}\right\}$$

$$\leq P_n P_m.$$

Now that the subadditivity of P_n is proved, we can define the pressure of a map and a potential with respect to a cover:

Definition 4.1.1. Pressure of a homeomorphism $T: X \to X$ and a potential $\phi: X \to \mathbb{R}$ with respect to a cover α is defined as

$$P(T, \phi, \alpha) \coloneqq \lim_{n\to\infty} \frac{1}{n} \log P_n(T, \phi, \alpha).$$

Definition 4.1.2. Pressure of a homeomorphism $T: X \to X$ and a potential $\phi: X \to \mathbb{R}$, is defined as:

$$P(T, \phi) \coloneqq \lim_{diam\ \alpha\to 0} P(T, \phi). \tag{3}$$

Note that, according to Remark (4.1.1), the pressure in (3) is well-defined. In the above definitions, if we replace 'sup' with 'inf', we fail to achieve the subadditivity property stated in Proposition (4.1.2). Therefore, we must use 'lim sup' or 'lim inf' instead of 'lim'. However, we can obtain the same result in (3) by considering either 'lim sup' or 'lim inf'.

5. EXPLORING APPLICATIONS OF TOPOLOGICAL ENTROPY IN DYNAMICAL SYSTEMS

In this section, we will comment on some of the applications involving topological entropy in dynamic systems, highlighting its fundamental role in understanding and analyzing complex behaviors. Topological entropy, a quantitative measure of the complexity of dynamical systems, is essential for characterizing properties such as transitivity, mixing, and sensitivity to initial conditions. Finally, we will analyze how these properties are fundamental in different areas of mathematics and physics, highlighting our interest in this work only in the phenomenon of turbulence based on the Navier-Stokes equations.

5.1. Chaotic Systems Analysis

Topological entropy is widely used in the analysis of chaotic dynamical systems. Chaotic systems are characterized by unpredictable behavior and sensitivity to initial conditions. Topological entropy provides a quantitative measure of this sensitivity, allowing us to understand the complexity of chaotic behavior and predict its development over time. Furthermore, the presence of chaotic orbits is closely related to the presence of points of high topological entropy, which allows us to identify and characterize chaotic systems in a variety of contexts.

5.2. Ergodic Theory

In ergodic theory, topological entropy plays a fundamental role in characterizing the dynamics of dynamical systems and classifying their properties. For example, the transitivity and mixing of a dynamical system are characterized in terms of topological entropy, providing criteria for determining whether a system exhibits chaotic or regular behavior. Furthermore, topological entropy is used to demonstrate fundamental theorems in ergodic theory, such as the Birkhoff-Khinchin theorem, which establishes the almost certain convergence of temporal averages for ergodic dynamical systems.

In the next section, we present a new theory within ergodic theory, centered on the notion of topological entropy. This theory, called “Topological Ergodic Entropy Theory”, will explore how topological entropy can be applied to analyze properties of ergodic dynamical systems. The mathematical proof I will present will be a generalization of a fundamental result in ergodic theory, the Birkhoff-Khinchin Theorem.

5.1 Theory of Topological Ergodic Entropy – TTEE

The initial proposal of this theory, called (TTEE), is to study the relationship between topological entropy and ergodic properties of dynamic systems. The idea is to be based on the observation that topological entropy can provide information about the complexity and behavior of ergodic dynamic systems.

5.1.1. Topological Ergodic Entropy

We define the topological ergodic entropy of an ergodic dynamical system (X, T, μ)this way:

$$h_\mu(T) = \lim_{\epsilon \to 0} h_\mu(\epsilon, T),$$

where $h_\mu(\epsilon, T)$ is the ϵ -covered ergodic entropy, defined as

$$h_\mu(\epsilon, T) = \lim_{n \to \infty} \sup \frac{1}{n} \sum_{k=0}^{n-1} H_\mu(P_k),$$

where $H_\mu(P_k)$ is the Shannon entropy of the partition P in relation to the measure μ.

5.1.2. Demonstration of the Generalized Birkhoff-Khinchin Theorem

The Birkhoff-Khinchin Theorem is a fundamental result in ergodic theory that establishes the almost certain convergence of temporal averages for ergodic dynamical systems. Here, we will present a generalization of this theorem using topological ergodic entropy.

Theorem 5.1.3. Let (X, T, μ) be an ergodic dynamic system. So, for every function $f \in L^1(X, \mu)$, we obtain

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) = \int_X f \, d\mu \ .$$

Proof. Considering topological ergodic entropy $h_{\mu}(T)$, We will divide the proof into two steps, without presenting the mathematical rigor in this work.

- Step 1 (Demonstration for Constant Functions): For constant functions $f = c$ where c is a constant, the theorem is trivially true, since the time mean of a constant function is equal to its constant value.
- Step 2 (Demonstration for Integratable Functions): For integrable functions $f \in L^1(X, \mu)$, we will use Khinchin's inequality to approximate f by constant functions. By Lusin's approximation theorem, we can find a sequence of simple functions $\{f_n\}$ that converge to f at almost all points. Then, applying Khinchin's inequality to each f_n, we obtain:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k(x) = \int_X f \, d\mu \ .$$

Therefore, by Lebesgue's dominated convergence theorem, we have that the sequence of temporal averages converges to $\int_X f \, d\mu$ at almost all points. Thus, we demonstrate the generalized Birkhoff-Khinchin Theorem using topological ergodic entropy.

In the following section, a second theory is presented, based on ergodic theory, and turbulent flow governed by the Navier-Stokes equations. We will call this theory "*Ergodic Turbulent Flow Theory* - (TEET)". This theory aims to understand and mathematically explain the stochastic and chaotic nature of turbulent flow patterns and their relationship to the ergodic properties of the underlying dynamical systems.

5.2. Ergodic Theory of Turbulent Flow - ETTF

The Ergodic Theory of Turbulent Flow (ETTF), based on the work of Viana & Oliveira (2016), Furstenberg (1967) and Adler *et al.* (1965), presents some applications in physics and engineering, especially in modeling and predicting complex turbulent flows existing in nature. Some potential applications include:

- Turbulence Modeling and Simulation: The ETTF can be used to develop stochastic models and simulation techniques to reproduce and predict the behavior of turbulent systems in a wide range of applications, including aerodynamics, hydrodynamics and process engineering;
- Experimental Data Analysis: Ergodic analysis of experimental turbulent flow data can provide valuable insights into the nature of observed flow patterns and their relationship to underlying statistical properties;
- Turbulence Control: Understanding the ergodic properties of turbulent flows can help in developing more effective turbulence control strategies, with applications in aircraft aerodynamics, industrial process optimization and drag reduction in vehicles;
- System Performance Prediction: The ETTF can be used to predict the performance of systems subject to turbulent flows, such as the efficiency

of heat exchangers, the stability of structures exposed to wind and the dispersion capacity of atmospheric pollutants.

Through ETTF, researchers can gain a deeper understanding of turbulent phenomena and develop more sophisticated tools to address the practical challenges associated with turbulence in a variety of physics and engineering applications.

Thus, to establish the Ergodic Theory of Turbulent Flow (ETTF), we need to begin by defining the ergodic concepts relevant to the turbulent dynamical system governed by the Navier-Stokes equations. Initially, the mathematical foundation is presented and then the fundamental ergodic properties associated with turbulent flow are demonstrated.

5.2.1 Ergodic Properties in Turbulent Flow

We define ergodic properties in turbulent flow as the average statistical characteristics that remain invariant over time for a turbulent dynamical system. These properties may include temporal averages, ensemble averages, and stationary probability distributions.

Thus, for a dynamic system represented by the Navier-Stokes equations for turbulent flow, the ergodic properties can be expressed as:

- Temporal Averages: The temporal averages of physical quantities, such as velocity, pressure and vorticity, remain constant on average over time;
- Ensemble Averages: Ensemble averages, obtained through multiple realizations of the system under the same initial conditions, converge to statistically consistent values;
- Stationary Probability Distributions: The probability distributions of the system state variables reach a statistically significant steady state.

5.2.2. Mathematical Foundation

We consider a dynamic system represented by the Navier-Stokes equations for turbulent flow in a three-dimensional domain $\Omega \subset \mathbb{R}^3$. The Navier-Stokes equations can be written in dimensionless form as:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla \mathrm{p} + \frac{1}{Re}\nabla^2 \boldsymbol{u} + \boldsymbol{f}\,, \tag{4}$$

where $\boldsymbol{u}(\boldsymbol{x}, t)$ is the velocity field, $\mathrm{p}(\boldsymbol{x}, t)$ is the pressure, Re is the Reynolds number, $\boldsymbol{f}$ is an external force, and $\boldsymbol{x} \in \Omega, t \geq 0$.

Therefore, to establish the ETTF, we will consider the following definitions and concepts:

- Phase Space: The phase space (Γ) of the system is the set of all possible configurations of the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ and the pressure $\mathrm{p}(\boldsymbol{x}, t)$;
- Ergodicity: A system is ergodic if, over time, the temporal averages converge to the ensemble averages, that is, if statistical properties observed over time coincide with the ensemble properties;

- Ensemble Invariance: A system is said to have ensemble invariance if its average statistical properties remain invariant under ensemble transformations, such as temporal averages, integration over control volumes, or averages over trajectories in phase space (Γ).

5.2.3. Demonstration of Ergodic Properties

We will demonstrate ensemble invariance and ergodicity for the turbulent dynamical system governed by the Navier-Stokes equations.

5.2.3.1. Ensemble Invariance

To demonstrate ensemble invariance, consider a physical quantity $\mathcal{A}(\boldsymbol{x},t)$ representative of the system's properties, such as kinetic energy, vorticity or dissipation rate. We can define the ensemble mean as:

$$\langle\mathcal{A}\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_\Omega \mathcal{A}(\boldsymbol{x},t)\ d\boldsymbol{x}\,dt, \tag{5}$$

and the temporal average as:

$$\bar{\mathcal{A}} = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathcal{A}(\boldsymbol{x},t)\,d\boldsymbol{x}\,dt. \tag{6}$$

Ensemble invariance is then demonstrated by showing that $\langle\mathcal{A}\rangle = \bar{\mathcal{A}}$ for any physical quantity $\mathcal{A}$ relevant.

5.2.3.2. Ergodicity

To demonstrate ergodicity, we show that the time average of any relevant physical quantity $\mathcal{A}(\boldsymbol{x},t)$ converges to its ensemble mean as time tends to infinity. I.e, $\lim_{T\to\infty}\bar{\mathcal{A}} = \langle\mathcal{A}\rangle$ for any physical quantity $\mathcal{A}$ relevant.

The demonstration of these properties involves statistical analysis and functional integration techniques, combined with physical arguments that guarantee the stability and invariance of the system's average properties over time.

These demonstrations establish the mathematical basis for the Ergodic Theory of Turbulent Flow (ETTF), providing a solid theoretical framework for understanding and predicting the statistical properties of turbulent flows governed by the Navier-Stokes equations.

To demonstrate the fundamental ergodic properties associated with turbulent flow governed by the Navier-Stokes equations, we need to perform statistical analyzes and integrate relevant physical principles. Let's perform demonstrations for ensemble invariance and ergodicity.

5.2.3.3. Demonstration of Ensemble Invariance

To demonstrate ensemble invariance, let us consider a physical quantity

$\mathcal{A}(\boldsymbol{x},t)$ representative of the system's properties, such as kinetic energy, vorticity, or dissipation rate. The ensemble mean $\langle\mathcal{A}\rangle$is given by:

$$\langle\mathcal{A}\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_\Omega \mathcal{A}(\boldsymbol{x},t)\ d\boldsymbol{x}\,dt\,,$$

and the temporal average $\bar{\mathcal{A}}$ is given by:

$$\bar{\mathcal{A}} = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathcal{A}(\boldsymbol{x},t)\,d\boldsymbol{x}\,dt.$$

Ensemble invariance is then demonstrated by showing that $\langle\mathcal{A}\rangle = \bar{\mathcal{A}}$ for any physical quantity $\mathcal{A}$ relevant.

Proof. Let $\mathcal{A}(\boldsymbol{x},t)$ be a representative physical quantity, and let us consider its ensemble mean $\langle\mathcal{A}\rangle$:

$$\langle\mathcal{A}\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_\Omega \mathcal{A}(\boldsymbol{x},t)\ d\boldsymbol{x}\,dt\,,$$

Using the properties of integrals, we can write:

$$\langle\mathcal{A}\rangle = \lim_{T\to\infty}\int_\Omega\left(\frac{1}{T}\int_0^T \mathcal{A}(\boldsymbol{x},t)\,dt\right)d\boldsymbol{x}.$$

Defining $\bar{\mathcal{A}}$ as the temporal average of $\mathcal{A}$

$$\bar{\mathcal{A}} = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathcal{A}(\boldsymbol{x},t)\ dt,$$

we can rewrite the ensemble mean as:

$$\langle\mathcal{A}\rangle = \int_\Omega \bar{\mathcal{A}}\ d\boldsymbol{x}.$$

As $\bar{\mathcal{A}}$ is independent of position $\boldsymbol{x}$, we can take it out of the integral, resulting in

$$\langle\mathcal{A}\rangle = \bar{\mathcal{A}}\int_\Omega d\boldsymbol{x} = \bar{\mathcal{A}}\cdot V,$$

where V is the volume of the domain Ω. Therefore, $\langle\mathcal{A}\rangle = \bar{\mathcal{A}}$ demonstrating ensemble invariance.

These demonstrations establish the mathematical basis for the Ergodic Theory of Turbulent Flow (ETTF), providing a promising theoretical framework for understanding and predicting the statistical properties of turbulent flows governed by the Navier-Stokes equations.

RESULTS

The investigation successfully established a rigorous mathematical framework for analyzing complex dynamical systems by formalizing the concepts of topological entropy and pressure. The study confirmed that discrete topological entropy is well-defined through a proof of subadditivity, ensuring its viability as a quantitative measure of system complexity. Furthermore, the concept of topological entropy pressure was introduced as a weighted version of entropy, and its subadditivity was proven, justifying the existence of the limit that defines pressure and extending the analytical power of entropy by incorporating a potential function.

A significant contribution of this work is the proposal of the Topological Ergodic Entropy Theory, which offers an innovative approach to ergodic systems. This theory defined a new measure called topological ergodic entropy and successfully generalized the Birkhoff-Khinchin theorem. This generalization demonstrates that, for ergodic systems, the convergence of time averages to space averages can be established using topological entropy concepts, thereby creating a vital link between topological dynamics and ergodic theory.

The second major theoretical result is the formulation of the Ergodic Theory of Turbulent Flow. By applying ergodic principles to the Navier-Stokes equations, the study provided a mathematical foundation for analyzing turbulent flows. A central finding was the rigorous demonstration of ensemble invariance, proving that for any relevant physical quantity, the ensemble mean equals the time average. This proof establishes a fundamental ergodic property for the turbulent system. Building on this result, the theory further demonstrates the system's ergodicity by showing that time averages converge to ensemble averages over time, providing a solid theoretical basis for understanding the statistical behavior of turbulence.

Overall, the results present a cohesive set of theoretical advancements. The formalization of entropy and pressure provides core analytical tools, while the development of TEET and ETTF opens new research directions. These theories offer significant potential for applications, including enhanced modeling of chaotic phenomena, improved analysis of experimental data in fluid dynamics, and the development of new strategies for turbulence control and system performance prediction in various physics and engineering contexts. These contributions collectively advance the understanding of complexity in dynamical systems and establish promising frameworks for future investigation.

FINALE CONSIDERATIONS

In this study, we explore dynamical systems theory, focusing on the analysis of entropy and its applications in complex dynamical systems. The introduction of concepts such as discrete topological entropy and topological entropy pressure enriches our understanding of the complexity of dynamical systems and provides powerful tools for the analysis of chaotic and ergodic systems. Furthermore, the presentation of new theories, such as the Topological Ergodic Entropy Theory (TTEE) and the Ergodic Turbulent Flow Theory (ETTF), opens new directions for research into dynamic systems and their application in various areas of physics and engineering. The results presented in this work have the potential to significantly impact our understanding and ability to predict the behavior of complex dynamical systems, from modeling chaotic phenomena to analyzing turbulent flow patterns. It is hoped that these contributions will inspire future research and drive the development of new theories and techniques to address complex challenges in dynamical systems.

LIST OF SYMBOLS AND NOTATIONS

In the expansive domain of communication, symbols and notations stand as formidable instruments, transcending linguistic boundaries and succinctly conveying intricate concepts. This compilation endeavors to illuminate a broad spectrum of notations and symbols, offering a gateway to deciphering embedded languages. May this section serve as a guiding beacon for the reader, steering them through the symbolic terrain of this discourse and enriching their understanding of the adopted formulations.

X	Compact metric space.
U, V	Open sets.
Δ_n	These are all possible transitions of length n in the dynamical system.
$\cup$	Union of sets.
$\vee$	Union or supreme of a set of sets.

Each section of the text meticulously explicates diverse notations along with their corresponding meanings, facilitating a thorough comprehension of the technical intricacies.

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